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Quadratic Splines

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Error bounds are obtained for quadratic splines satisfying a mean averaging condition with respect to a nonnegative measure.

1. INTRODUCTION

Recently Schoenberg [5] and de Boor [1] have considered even degree splines whose integral means between knots agree with the same means of a given function. Schoenberg has brought out the interest of this kind of mean averaging condition in statistical problems. Earlier studies on even degree splines have considered splines which interpolate a given function at the midpoints of knots. More recently, without making any assumption on the partition, Marsden [4] has determined the error bounds for quadratic splines which interpolate a given function at the midpoints of knots. This result is unlike some results on cubic spline interpolation which depend on the nature of the partition. Our aim is to show that error bounds similar to those of Marsden hold for quadratic splines satisfying mean averaging conditions with respect to a nonnegative measure. It is of some interest to mention that Varga [7] has considered error bounds for splines satisfying conditions involving functionals using a different approach.

In Section 2 we study the problem of existence and uniqueness of a quadratic spline satisfying the mean averaging condition (2.1). Section 3 deals with a special case not covered by Theorem 1 of Section 2. We also give a best approximation property of the interpolatory quadratic spline when the number of knots is odd and the interpolant is periodic. In Section 4, we obtain explicit error bounds when the interpolant is 1-periodic and is either continuous or $\in C^1$ or C^2.

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2. Existence and Uniqueness

Let $\Delta = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ be a partition of $[0, 1]$ and let $\mathrm{Sp}(m, \Delta) = \{s(x): s(x) \in C^{m-1}[0, 1], \quad s(x) \in \pi_m \quad \text{for} \quad x \in (x_i, x_{i+1}), \quad i = 0, 1,..., n-1\}$, where m is any positive integer. We propose

Problem A. *Let f be a 1-periodic locally integrable function with respect to a nonnegative measure $d\mu$. Find $s(x) \in \mathrm{Sp}(2, \Delta)$ where $s(0) = s(1)$, $s'(0) = s'(1)$ and*

$$\int_{x_i}^{x_{i+1}} (f(x) - s(x))\, d\mu = 0, \qquad i = 0, 1,..., n-1. \tag{2.1}$$

If $\mu(x) = x$, condition (2.1) has been studied by Schoenberg [5] and also by Demko and Varga [2]. If $\mu(x)$ is a step function with jumps of one at the midpoints of knots, then (2.1) reduces to

$$f\left(\frac{x_i + x_{i+1}}{2}\right) - s\left(\frac{x_i + x_{i+1}}{2}\right) = 0, \qquad i = 0, 1,..., n-1, \tag{2.1a}$$

which was considered by Marsden [4].

By a suitable choice of $d\mu$, it is possible to reduce (2.1) into

$$s(x_i) + s(x_{i+1}) = f(x_i) + f(x_{i+1}), \qquad i = 0, 1,..., n-1. \tag{2.1b}$$

We shall return to this special case in Section 3.

The number of parameters in $s(x)$ is $n + 2$ which agrees with the number of linear conditions given by (2.1) and the assumption of periodicity.

We shall now need two kinds of representation for the spline $s(x)$ of the problem. The first one is in terms of $s'(x_j) \equiv M_j$, $j = 0, 1,..., n-1$, and the second one is in terms of $s(x_j) \equiv s_j$, $j = 0, 1,..., n-1$.

Let us first find a representation of $s(x)$ in terms of M_j. Then $s'(x)$ is linear in $x_{j-1} \leqslant x \leqslant x_j$, so that setting $h_j = x_j - x_{j-1}$, we have for $x \in [x_{j-1}, x_j]$,

$$s(x) = -M_{j-1}\frac{(x_j - x)^2}{2h_j} + M_j\frac{(x - x_{j-1})^2}{2h_j} + c_j, \qquad j = 1,.., n. \tag{2.2}$$

Here c_j's are constants determined by the continuity requirement of $s(x)$. Hence

$$M_j(h_j + h_{j+1}) = 2(c_{j+1} - c_j), \qquad j = 0, 1,..., n-1. \tag{2.3}$$

Set

$$F_j = \int_{x_{j-1}}^{x_j} f\, d\mu, \qquad H_j = \int_{x_{j-1}}^{x_j} d\mu, \qquad K_j = \int_{x_{j-1}}^{x_j} (x - x_{j-1})(x_j - x)\, d\mu,$$

and

$$A_j^{(\nu)} = \int_{x_{j-1}}^{x_j} (x_j - x)^\nu \, d\mu, \qquad B_j^{(\nu)} = \int_{x_{j-1}}^{x_j} (x - x_{j-1})^\nu \, d\mu, \qquad \nu = 1, 2,$$

for $j = 1,..., n$.

If $s(x)$ satisfies (2.1), we get from (2.2)

$$c_j H_j - \frac{M_{j-1}}{2h_j} A_j^{(2)} + \frac{M_j}{2h_j} B_j^{(2)} = F_j, \qquad j = 1,..., n. \tag{2.4}$$

Suppose $H_j \neq 0$ for all j. Then from (2.3) and (2.4), we have the following system of equations in the parameters M_j:

$$\begin{aligned} M_{j-1} \cdot \frac{A_j^{(2)}}{2h_j H_j} + \frac{1}{2} M_j \left[(h_j + h_{j+1}) - \frac{B_j^{(2)}}{h_j H_j} - \frac{A_{j+1}^{(2)}}{h_{j+1} H_{j+1}} \right] \\ + M_{j+1} \cdot \frac{B_{j+1}^{(2)}}{2h_{j+1} H_{j+1}} = \frac{F_{j+1}}{H_{j+1}} - \frac{F_j}{H_j}, \qquad j = 1,..., n. \end{aligned} \tag{2.5}$$

The coefficients of M_{j-1}, M_j, M_{j+1} are easily seen to be nonnegative. A sufficient condition for the existence and uniqueness of the solution of the system of equations (2.5) and hence of the Problem A is that the matrix of the system (2.5) is irreducibly diagonally dominant. The difference of the coefficient of M_j and of the sum of the coefficients of M_{j-1} and M_{j+1} is easily seen to be

$$\frac{K_j}{h_j H_j} + \frac{K_{j+1}}{h_{j+1} H_{j+1}},$$

which is certainly $\geqslant 0$. Since the matrix of system (2.5) is irreducible, we shall only require that

$$\int_{x_{j-1}+}^{x_j-} d\mu > 0 \qquad \text{for some } j. \tag{2.6}$$

We have thus proved

THEOREM 1. *If (2.6) holds for some j, and $H_j > 0$ for every j, then there exists a unique $s(x) \in \mathrm{Sp}(2, \Delta)$ which satisfies the conditions of Problem* A.

We shall now find a representation of $s(x)$ in terms of its values at the nodes. Indeed, for $x_{j-1} \leqslant x \leqslant x_j$, we have

$$s(x) = \frac{x - x_{j-1}}{h_j} s_j + \frac{x_j - x}{h_j} s_{j-1} + D_j \cdot (x - x_{j-1})(x_j - x),$$

where the constants D_j are to be determined from the continuity of $s'(x)$ and from (2.1). This leads to the following system of equations:

$$\begin{aligned} \frac{s_j - s_{j-1}}{h_j} - D_j h_j &= \frac{s_{j+1} - s_j}{h_{j+1}} + D_{j+1} h_{j+1}, \qquad j = 1,\dots, n, \\ \frac{s_j}{K_j} B_j^{(1)} + \frac{s_{j-1}}{K_j} A_j^{(1)} + D_j h_j &= \frac{F_j h_j}{K_j}, \qquad j = 1,\dots, n. \end{aligned} \tag{2.7}$$

Eliminating D_j's from the system (2.7), we have the system of equations to determine s_j:

$$s_{j-1}\left\{\frac{A_j^{(1)}}{K_j} - \frac{1}{h_j}\right\} + s_j\left\{\frac{1}{h_j} + \frac{1}{h_{j+1}} + \frac{B_j^{(1)}}{K_j} + \frac{A_{j+1}^{(1)}}{K_{j+1}}\right\} + s_{j+1}\left\{-\frac{1}{h_{j+1}} + \frac{B_{j+1}^{(1)}}{K_{j+1}}\right\} = \frac{F_{j+1}h_{j+1}}{K_{j+1}} + \frac{F_j h_j}{K_j} \qquad (j = 1, 2,\dots, n). \tag{2.8}$$

It is easy to see that elements of the matrix of this system are nonnegative. Under the condition of Theorem 1, we know that $s(x)$ exists and is unique. Hence system (2.8) has a unique solution. However, we would like to know when the matrix of (2.8) is diagonally dominant. A sufficient condition for this is that $d\mu$ be symmetric in each subinterval (x_{j-1}, x_j) about the midpoint.

3. A Special Case

In the special case of (2.1b), it is easy to see that condition (2.6) is not satisfied. However it is still possible to solve Problem A in this case. We seek the spline $S(x) \in S(2, \Delta)$ such that

$$S(x_\nu) + S(x_{\nu+1}) = f_\nu + f_{\nu+1}, \qquad \nu = 0, 1,\dots, n-1, \tag{3.1}$$

where $f_\nu = f(x_\nu)$. Denoting again $S'(x_\nu)$ by M_ν, we get as above the following system of equations in M_ν:

$$h_\nu M_{\nu-1} + (h_\nu + h_{\nu+1}) M_\nu + h_{\nu+1} M_{\nu+1} = 2(f_{\nu+1} - f_{\nu-1}), \qquad \nu = 1,\dots, n, \tag{3.2}$$

where $M_n = M_0$, $f_n = f_0$. The system (3.2) is not diagonally dominant. However, we first rewrite (3.2) in the form

$$h_\nu(M_{\nu-1} + M_\nu) - 2(f_\nu - f_{\nu-1}) = -[h_{\nu+1}(M_\nu + M_{\nu+1}) - 2(f_{\nu+1} - f_\nu)], \qquad \nu = 1,\dots, n,$$

and if n is odd, we get

$$h_\nu(M_{\nu-1} + M_\nu) = 2(f_\nu - f_{\nu-1}), \tag{3.3}$$

whence we have for n odd

$$M_\nu = \sum_{k=0}^{n-1} (-1)^k \frac{f_{k+\nu+1} - f_{k+\nu}}{h_{k+\nu+1}}.$$

As in Section 2, we have for $x_{\nu-1} \leqslant x \leqslant x_\nu$,

$$S(x) = -M_{\nu-1} \frac{(x_\nu - x)^2}{2h_\nu} + M_\nu \cdot \frac{(x - x_{\nu-1})^2}{2h_\nu} + c_\nu, \tag{3.4}$$

where M_ν's and c_ν's are connected by the following relations because of (3.1):

$$-\frac{M_{\nu-1}h_\nu}{2} + 2c_\nu + \frac{M_\nu h_\nu}{2} = f_{\nu-1} + f_\nu, \qquad \nu = 1,\dots, n. \tag{3.5}$$

Hence it follows from (3.3), (3.4) and (3.5) that $S(x_\nu) = f_\nu$, $\nu = 1,\dots, n$. We have thus proved

THEOREM 2. *For every* 1-*periodic function* $f(x)$ *there exists a unique quadratic spline* $S(x) \in S_p(2, \Delta)$ *satisfying* (3.1) *if and only if n is odd. Moreover,* $S(x)$ *interpolates* $f(x)$ *at the nodes.*

Remark 1. If (3.1) is replaced by

$$S(x_\nu) + \alpha S(x_{\nu+1}) = f_\nu + \alpha f_{\nu+1}, \qquad \nu = 1,\dots, n, \tag{3.1a}$$

then the conclusion of Theorem 2 is still valid if $\alpha \neq -1$.

We shall now show that the spline $S(x)$ of Theorem 2 has a best approximation property with respect to the functional $L(f)$ where

$$L(f) = \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} \{f'(x) + f'(x_{j-1} + x_j - x)\}^2\, dx.$$

More precisely we shall prove

THEOREM 3. *Let* $\Delta = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ *be a partition for a given odd integer n. Let* $f(x)$ *be absolutely continuous on* $[0, 1]$ *with* $f'(x) \in L^2[0, 1]$. *If* S_f *is the interpolatory quadratic spline of Theorem* 2, *then*

$$L(f - S) \geqslant L(f(x) - S_f(x)), \tag{3.6}$$

where $S(x)$ *is any* 1-*periodic spline* $\in \mathrm{Sp}(2, \Delta)$. *Moreover, equality will hold only if* $S(x) = S_f(x) + c$ *where c is an arbitrary constant.*

Proof. We shall need the identity

$$L(F(x) - S_F(x)) = L(F) - L(S_F(x)). \tag{3.7}$$

The proof of this follows on rewriting $L(F - S_F)$ as

$$L(F) - L(S_F) - 2 \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} [F'(x) + F'(x_{j-1} + x_j - x) - S_F'(x)$$
$$- S_F'(x_{j-1} + x_j - x)] \cdot [S_F'(x) + S_F'(x_{j-1} + x_j - x)]\, dx$$

and on observing that the last sum vanishes on integration by parts.

We observe that if $f(x)$ is 1-periodic and if $S(x)$ is any 1-periodic quadratic spline $\in \mathrm{Sp}(2, \Delta)$, then

$$S_{f-S} = S_f - S.$$

Replacing $F(x)$ in (3.7) by $f(x) - S(x)$, we get

$$L(f - S) = L(S_{f-S}) + L(f - S_f),$$

which proves inequality (3.6). In case of equality in (3.6), we have

$$L(S_{f-S}) = 0.$$

Let $\tilde{M}_j$ denote the slope of the quadratic spline S_{f-S} at x_j. Then from the definition of L, we see that

$$\tilde{M}_{j-1} + \tilde{M}_j = 0, \qquad j = 1,..., n$$

and since n is odd, $\tilde{M}_j = 0$ for all j. Hence $S_{f-S} \equiv S_f - S = c$, which completes the proof.

4. Error Estimates

In the sequel $\omega(f; h)$ will denote the modulus of continuity of f.

(a) Let f be 1-periodic and $\in C^2[0, 1]$. Set $e(x) = s(x) - f(x)$ where $s(x)$ is the quadratic interpolatory spline of Theorem 1 and let $e_i^{(\nu)} = e^{(\nu)}(x_i)$, $\nu = 0, 1, 2$. We shall now prove

THEOREM 4. *The following error bounds hold*:

$$\| e \| \leqslant \bar{\Delta}(L(\bar{\Delta}) + (\bar{\Delta}/2))\, \omega(f''; \bar{\Delta}), \tag{4.1}$$

$$\| e' \| \leqslant (L(\bar{\Delta}) + (\bar{\Delta}/2))\, \omega(f''; \bar{\Delta}), \tag{4.2}$$

$$\| e'' \| \leqslant (1 + (L(\bar{\Delta})/\underline{\Delta}))\, \omega(f''; \bar{\Delta}), \tag{4.3}$$

where $\bar{\Delta} = \max_i h_i$, $\underline{\Delta} = \min_i h_i$ *and*

$$L(\bar{\Delta}) = \max_{\nu} \{A_\nu^{(2)} h_\nu / 2K_\nu ,\, B_\nu^{(2)} h_\nu / 2K_\nu\}. \tag{4.4}$$

Proof. From (2.5), we can easily write the system of equations for e_i' as

$$\begin{aligned} &e'_{j-1} \cdot \frac{A_j^{(2)}}{2h_jH_j} + \frac{1}{2} e_j' \left\{(h_j + h_{j+1}) - \frac{B_j^{(2)}}{h_jH_j} - \frac{A_{j+1}^{(2)}}{h_{j+1}H_{j+1}}\right\} \\ &\quad + e'_{j+1} \cdot \frac{B_{j+1}^{(2)}}{2h_{j+1}H_{j+1}} \\ &= \frac{F_{j+1}}{H_{j+1}} - \frac{F_j}{H_j} - f'_{j-1} \frac{A_j^{(2)}}{2h_jH_j} - \frac{1}{2} f_j' \left\{(h_j + h_{j+1}) - \frac{B_j^{(2)}}{h_jH_j} - \frac{A_{j+1}^{(2)}}{h_{j+1}H_{j+1}}\right\} \\ &\quad - f'_{j+1} \cdot \frac{B_{j+1}^{(2)}}{2h_{j+1}H_{j+1}}. \end{aligned} \tag{4.5}$$

Since $f(x) = f_j + (x - x_j) f_j' + ((x - x_j)^2/2) f''(\xi_j)$, and since $f'(x) = f_j' + (x - x_j) f''(\eta_j)$, where ξ_j, η_j lie between x_j and x, the right side of (4.5) becomes

$$\frac{f''(\xi_{j+1})}{2} \frac{B_{j+1}^{(2)}}{H_{j+1}} - \frac{f''(\xi_j)}{2} \frac{A_j^{(2)}}{H_j} + f''(\eta_j) \cdot \frac{A_j^{(2)}}{2H_j} - f''(\eta_{j+1}) \frac{B_{j+1}^{(2)}}{2H_{j+1}}.$$

By the classical argument based on the diagonal dominance property, we get

$$\max_j | e_j' | \leqslant L(\bar{\Delta}) \cdot \omega(f''; \bar{\Delta}).$$

Since $e'(x)$ is linear in each interval (x_{i-1}, x_i), the rest of the argument follows the reasoning in [3, p. 245].

(b) Let $f \in C[0, 1]$ and let f be 1-periodic. Then we can prove

THEOREM 5. *The following error bounds hold*:

$$\| e \| \leqslant \bar{\Delta}(1 + \Lambda(\bar{\Delta}))\, \omega(f'; \bar{\Delta}), \tag{4.6}$$

$$\| e' \| \leqslant (1 + \Lambda(\bar{\Delta}))\, \omega(f'; \bar{\Delta}), \tag{4.7}$$

where

$$\Lambda(\bar{\Delta}) = \max_\nu \left\{\frac{2h_\nu A_\nu^{(1)} + A_\nu^{(2)}}{2K_\nu}, \frac{2h_\nu B_\nu^{(1)} + B_\nu^{(2)}}{2K_\nu}\right\}.$$

The proof again begins with the system of equations (4.5) and follows the same lines as that of Theorem 3 with suitable modifications.

(c) If $f \in C[0, 1]$ and is 1-periodic, we have to use the system of equations (2.8). In order to be able to use the diagonal dominance property, we shall assume that $d\mu$ is symmetric in each subinterval (x_{j-1}, x_j) about the midpoint. Then we can prove

THEOREM 6. *When f is 1-periodic and $\in C[0, 1]$, we have*

$$\| e \| \leqslant (1 + \max_j (h_j^2 H_j/K_j))\, \omega(f; \bar{\Delta}/2). \tag{4.8}$$

The proof is based on the system of equations (2.8) and follows the same lines as above with suitable modifications. In particular cases when $d\mu$ is some specific measure, it may be possible to reduce the constant on the right side in (4.8). More precisely, in the case of interpolation at the midpoints, we can get the same constant as in [4, Theorem 2.1].

EXAMPLE. If $d\mu = dx$, then $L(\bar{\Delta}) = \bar{\Delta}$ in Theorem 4 and $\Lambda(\bar{\Delta}) = 4$ in Theorem 5 and in formula (4.8), we get $\| e \| \leqslant 7\omega(f; \bar{\Delta}/2)$.

Remark 2. It is easy to construct an example of a measure $d\mu$ which is not symmetric in any subinterval about its midpoint such that the system of equations do not have a diagonally dominant matrix. More precisely, if we interpolate a continuous 1-periodic function by quadratic splines at $\alpha x_{\nu-1} + (1 - \alpha) x_\nu$ for a fixed positive small α, $(\nu = 1,..., n)$, then the estimate for the error $e(x)$ depends on the relative size of successive intervals.

Remark 3. The best approximation property of Theorem 3 can be easily generalized to general even degree splines with an odd number of knots which interpolate a given periodic function at the knots.

Remark 4. The estimates (4.1), (4.2) in Theorem 4, (4.6), (4.7) in Theorem 5, and (4.8) in Theorem 6 do not depend on the mesh ratio for a large number of measures. It may be of interest to point out that the smallest value of the constant $\max_j h_j^2 H_j / K_j$ which appears in (4.8) of Theorem 6 takes its minimum value when we deal with interpolation at the midpoints.

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